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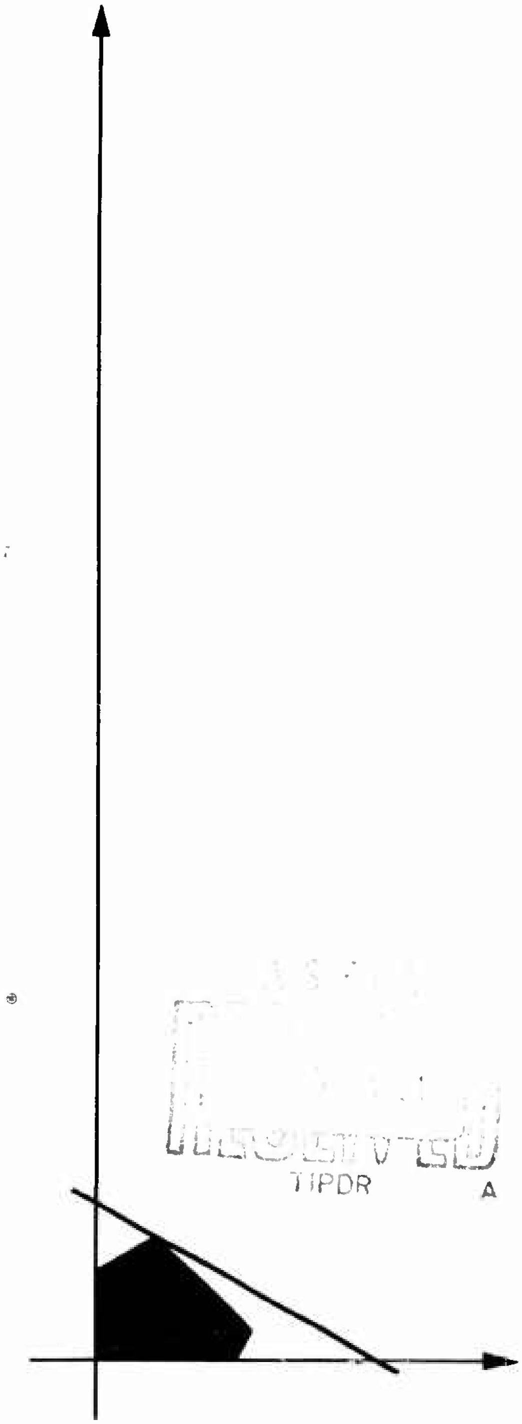
DUALITY IN HOMOGENEOUS PROGRAMMING

by

E. Eisenberg



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DUALITY IN HOMOGENEOUS PROGRAMMING

The problem of maximizing a concave function subject to linear constraints does not have a dual, as is the case in linear programming, in which primal optimizing variables do not appear. As a special case of our principal result it will follow that such a dual does indeed exist whenever the objective function is also homogeneous.

In the linear case we are given an $m \times n$ matrix A and vectors* $a \in R^n$, $b \in R^m$. The feasibility sets X and Y are defined by: $X = R^m_+ \cap \{x \mid xA \leq a\}$, $Y = R^n_+ \cap \{y \mid Ay \geq b\}$. Since $xA \leq a$ if and only if $xAy \leq ay$ for all $y \in R^n_+$ (and similarly for $Ay \geq b$), we may write:

$$\text{(1)} \qquad \begin{aligned} X &= R^m_+ \cap \left\{x \mid xAy \leq \psi(y) \quad \text{all } y \in R^n_+\right\} \\ Y &= R^n_+ \cap \left\{y \mid xAy \geq \phi(x) \quad \text{all } x \in R^m_+\right\} \end{aligned}$$

where $\psi(y) = ay$ and $\phi(x) = bx$.

A fundamental theorem of linear programming (see, e.g., [3] and [5]) states that if X and Y are both non-empty then

$$\text{(2)} \qquad \max_{x \in X} \phi(x), \ \min_{y \in Y} \psi(y) \ \text{exist and are equal}.$$

We propose to demonstrate that (2) holds for larger class of triples (A, ϕ, ψ).

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* R^m denotes the set of all real m-tuples. If $u, v \in R^m$ then $u \leq v$ means that the inequality holds for each component. In particular, $R^m_+ = R^m \cap \{x \mid x \geq 0\}$. If M is a $p \times q$ matrix and N is a $q \times t$ matrix then $M N$ represents the usual matrix product. To simplify notation, the same symbol is used for both a column vector and its transpose; the meaning will, in any case, be clear from the context.

Assumption A_1. Let $\phi : R_+^m \to R$, $\psi : R_+^n \to R$ be positively homogeneous*, continuous, concave and convex respectively.

Let us first show that A_1 does not guarantee that (2) holds when X and Y are non-empty. If $m = 2$, $n = 1$ and $A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\phi(x) = \phi(\xi, \eta) = \dfrac{\xi \eta}{\xi + \eta}$, $[\phi(0) = 0]$, $\psi(y) = y$ then A_1 is satisfied and $X = R_+^2 \cap \{(\xi, \eta) | \eta \le 1\}$, $Y = R_+ \cap \{y | y \ge 1\}$ are non-empty. Thus $\min_{y \in Y} \psi(y) = 1$, but if $\eta \le 1$ then $\phi(\xi, \eta) < 1$, although $\sup_{x \in X} \phi(x) = 1$, hence $\max_{x \in X} \phi(x)$ does not exist.

The situation just illustrated cannot occur if the following holds:

Assumption A_2.

i) If $x \in R_+^m$, $xA \le 0$, $\phi(x) \ge 0$ then $x = 0$

ii) If $y \in R_+^n$, $Ay \ge 0$, $\psi(y) \le 0$ then $y = 0$

One sees immediately that (i) is violated in the preceding example, for let $x = (1, 0)$ then $xA = 0$ and $\phi(x) = 0$.

Before proving that if A_1 and A_2 hold then so does (2), we require the following lemma which specializes to homogeneous functions the well-known fact that a concave function is the infimum of its supports. The proof is presented here for the sake of completeness.

Lemma 1. Let ϕ be as in assumption A_1, consider $T = R^m \cap \{t | tx \ge \phi(x) \text{ all } x \in R_+^m\}$, then T is non-empty, and $\phi(x) = \inf_{t \in T} tx$, for all $x \in R_+^m$.

Proof. Let $C = \{(x, \lambda) | x \in R_+^m, \lambda \le \phi(x)\}$ then C is a closed convex cone. Now if $x_0 \in R_+^m$, $\mu > 0$, then $(x_0, \mu + \phi(x_0)) \notin C$, whence (see [2], Theorem 1)

* A function $f : C \to R^q$, where $C \subset R^p$ is a cone, is positively homogeneous providing $f(\lambda x) = \lambda f(x)$ for all $x \in C$ and $\lambda \in R_+$.

there exist $t \in R^m$ and $a \in R$ such that $tx_0 - a[\mu + \phi(x_0)] < 0 \leq tx - a\lambda$ all $(x, \lambda) \in C$.

It then follows that $a > 0$, so that (dividing by a) we may assume $a = 1$, but then $t \in T$. Reiterating, if $x_0 \in R^m_+$, $\mu > 0$ then $\exists t \in T$ such that:

$$tx_0 - \mu \leq \phi(x_0) \leq tx_0$$

giving the desired result. We are now able to prove:

<u>Theorem 1</u>. If assumptions A_1 and A_2 hold then (2) holds.

<u>Proof</u>. Let

$$\text{(3)} \qquad \begin{aligned} S &= R^n \cap \{s \mid sy \leq \psi(y), \quad \text{all } y \in R^n_+\} \\ T &= R^m \cap \{t \mid tx \geq \phi(x), \quad \text{all } x \in R^m_+\} \end{aligned}$$

Then S and T are convex sets; now consider the system of inequalities:

$$\text{(4)} \qquad \begin{aligned} &(x, y, s, t) \in R^m_+ \times R^n_+ \times S \times T \\ &s - xA > 0 \\ &-t + Ay > 0 \\ &\phi(x) - \psi(y) > 0 \end{aligned}$$

If (4) has a solution x, y, s, t then

$$\psi(y) < \phi(x) \leq tx \leq xAy \leq sy \leq \psi(y)$$

which is a contradiction. Thus (see [2], Theorem 1) there exist $x_0 \in R^m_+$, $y_0 \in R^n_+$, $\lambda \in R_+$, not all zero and such that $(s - xA)y_0 + x_0(Ay - t) + \lambda[\phi(x) - \psi(y)] \leq$ ≤ 0 for all $x \in R^m_+$, $y \in R^n_+$, $s \in S$, $t \in T$. From the homogeneity and continuity of ϕ and ψ it then follows that:

$$xAy_0 \geq \lambda\phi(x) \quad , \qquad \text{all } x \in R_+^m$$

$$x_0Ay \leq \lambda\psi(y) \quad , \qquad \text{all } y \in R_+^n$$

$$sy_0 \leq tx_0 \quad , \qquad \text{all } s \in S,\ t \in T$$

The last condition together with Lemma 1 imply:

$$\psi(y_0) \leq \phi(x_0)$$

Now if $\lambda = 0$ then either $x_0 \neq 0$ or $y_0 \neq 0$ and $Ay_0 \geq 0$, $x_0A \leq 0$. Suppose $x_0 \neq 0$, if $y_0 \neq 0$ the argument is analogous, then by A_2(i) we have $\phi(x_0) < 0$, whence $\psi(y_0) < 0$ and $y_0 \neq 0$, contradicting A_2(ii). Thus $\lambda > 0$ and, dividing all inequalities by λ, we may assume $\lambda = 1$. This tells us that $x_0 \in X$, $y_0 \in Y$ and $\phi(x_0) \leq x_0Ay_0 \leq \psi(y_0) \leq \phi(x_0)$. So that if $x \in X$, $y \in Y$ then

$$\phi(x) \leq xAy_0 \leq \psi(y_0) = \phi(x_0)$$

$$\psi(y) \geq x_0Ay \geq \phi(x_0) = \psi(y_0)$$

proving the theorem.

In case ϕ and ψ are linear-homogeneous then it is true that $\max_{x \in X} \phi(x)$ exists if and only if $\min_{y \in Y} \psi(y)$ exists, in which case they are equal. As above, this statement is not always true under assumption A_1; however, we show:

Theorem 2.

I) If A_1 and A_2(ii) hold, and $\max_{x \in X} \phi(x)$ exists then (2) holds.

II) If A_1 and A_2(i) hold, and $\min_{y \in Y} \psi(y)$ exists then (2) holds.

We prove (I), the proof of (II) is similar. Suppose that $x_0 \in X$ and $\phi(x_0) =$ $= \max_{x \in X} \phi(x)$, then the system:

$$(6) \qquad \begin{aligned} &(x, s) \in R_+^m \times S \\ &s - xA > 0 \\ &\phi(x) - \phi(x_0) > 0 \end{aligned}$$

has no solution. Thus (see [2], Theorem 1) there exist $y_0 \in R_+^n$, $\lambda \in R_+$, not both zero and such that

$sy_0 - xAy_0 + \lambda[\phi(x) - \phi(x_0)] \leq 0$ for all $x \in R_+^m$, $s \in S$. From the homogeneity of ϕ and Lemma 1 it then follows that

$$(7) \qquad \begin{aligned} &xAy_0 \geq \lambda\phi(x) \quad , \qquad \text{for all } x \in R_+^m \\ &\psi(y_0) \leq \lambda\phi(x_0) \end{aligned}$$

Now, if $\lambda = 0$ then $y_0 \neq 0$ and $Ay_0 \geq 0$, $\psi(y_0) \leq 0$, contradicting A_2(ii). It may then be assumed that $\lambda > 0$ and, in fact, that $\lambda = 1$ (replacing y_0 by λy_0). Thus, from (7), $y_0 \in Y$, and for any $y \in Y$ we have:

$$\psi(y_0) \leq \phi(x_0) \leq x_0 Ay \leq \psi(y) ,$$

i.e., $$\psi(y_0) = \min_{y \in Y} \psi(y) = \phi(x_0) .$$ q.e.d.

It should be remarked that if A_1 holds then (i) and (ii) of assumption A_2 are equivalent to (i)' and (ii)' respectively of:

<u>Assumption A_2'.</u>

i)' $\exists\, y_0 \in R_+^n \ni xAy_0 > \phi(x)$ all $x \in R_+^m$, $x \neq 0$

ii)' $\exists\, x_0 \in R_+^m \ni x_0 Ay < \psi(y)$ all $y \in R_+^n$, $y \neq 0$

These in turn are equivalent to the familiar conditions that X, Y have non-empty interiors. To see, for instance, that (i) and (i)' are equivalent it suffices to show that (i) implies (i)' since the implication in the other direction is trivial. Assuming (i)' false, the system

(8) $$(y,t) \in R^n_+ \times T$$
$$Ay - t > 0$$

has no solution, whence (see [2], Theorem 1) there is an $x \in R^m_+$, $x \neq 0$, and such that $xAy \leq tx$ for all $y \in R^n_+$ and $t \in T$. Thus $xA \leq 0$ and (using Lemma 1) $\phi(x) \geq 0$, contradicting (i). To return to our remark about maximizing a concave homogeneous and continuous function $\phi: R^m_+ \to R$, subject to the inequalities $x \geq 0$ and $xA \leq a$, the dual is then: minimize ay subject to $y \in Y$. Conditions (i) and (ii)' become:

$x \in R^m_+$, $x \neq 0$, $xA \leq 0$, $\phi(x) \geq 0$ has no solution; and

$x \in R^m_+$, $xA < a$ has a solution; respectively.

Also, since $y \in Y$ providing $y \geq 0$ and $Ay \geq t$ for some support t of ϕ, we may characterize Y by means of the gradient of ϕ.

Results similar to Theorems 1 and 2 can be shown to hold under other and somewhat less restrictive assumptions; the duality theorems of <u>linear</u> programming then turn out to be special cases of these theorems (3, 4 and 5).

<u>Henceforth we assume that A_1 holds</u> and consider the sets:

$$K_1 = R^{m+n+1} \cap \{(\bar{x}, \bar{y}, \lambda) \mid \exists\ s \in S,\ t \in T,\ x \in R^m_+,\ y \in R^n_+ \text{ and }$$
$$\bar{x} \geq t - Ay,\ \bar{y} \leq s - xA,\ \lambda \leq \phi(x) - \psi(y)\},$$

$$K_2 = R^{n+1} \cap \{(y, \lambda) \mid \exists\ s \in S,\ x \in R^m_+, \text{ and } \bar{y} \leq s - xA,\ \lambda \leq \phi(x)\},$$

$$K_3 = R^{m+1} \cap \{(x, \lambda) \mid \exists\ t \in T,\ y \in R^n_+, \text{ and } \bar{x} \geq t - Ay,\ \lambda \geq \psi(y)\}.$$

The sets K_1, K_2, and K_3 are readily seen to be convex (because ψ and $-\phi$ are convex); furthermore, if ϕ and ψ are linear then K_1, K_2 and K_3 are also closed sets. Of course any (or all) of K_1, K_2, K_3 may be closed without either ϕ or ψ being linear. Thus it is important to know the following:

Theorem 3.

If K_1 is closed and X, Y are both non-empty, then (2) holds.

Theorem 4.

If K_3 is closed and $\max\limits_{x \in X} \phi(x)$ exists, then (2) holds.

Theorem 5.

If K_2 is closed and $\min\limits_{y \in Y} \psi(y)$ exists, then (2) holds.

Proof of Theorem 3: If the point $(\bar{x}, \bar{y}, \lambda) = 0$ is in K_1 then (2) obviously holds, suppose $0 \notin K_1$. Since K_1 is convex and closed, there exist (see [3]) $x_0 \in R^m$, $y_0 \in R^n$, $\lambda_0 \in R$, $\alpha \in R$, such that:

$$0 < \alpha \leq x_0\bar{x} - y_0\bar{y} - \lambda_0\lambda, \qquad \text{all } (\bar{x}, \bar{y}, \lambda) \in K_1 \quad .$$

Since S and T are non-empty (see Lemma 1), and since $(\bar{\bar{x}}, \bar{\bar{y}}, \lambda') \in K_1$ whenever there exist $(\bar{x}, \bar{y}, \lambda) \in K_1$ such that $\bar{\bar{x}} \geq \bar{x}$, $\bar{\bar{y}} \leq \bar{y}$ and $\lambda' \leq \lambda$, it follows that $x_0 \geq 0$, $y_0 \geq 0$, and $\lambda_0 \geq 0$. Also,

$$0 < \alpha \leq x_0 (t - Ay) - (s - xA)y_0 - \lambda_0 [\phi(x) - \psi(y)] ,$$

$$\text{all } (s, t, x, y) \in S \times T \times R^m_+ \times R^n_+$$

From the homogeneity of ϕ and ψ and Lemma 1 it then follows that:

$$0 < \alpha \leq \phi(x_0) - \psi(y_0)$$

$$x_0Ay \leq \lambda_0 \psi(y) \qquad \text{all } y \in R^n_+$$

$$xAy_0 \geq \lambda_0 \phi(x) \qquad \text{all } x \in R^m_+$$

Thus, $\lambda_0 \psi(y_0) \geq x_0 A y_0 \geq \lambda_0 \phi(x_0) \geq \lambda_0 \alpha + \lambda_0 \psi(y_0)$, and $\lambda_0 = 0$. But then $x_0 A \leq 0$ and $Ay_0 \geq 0$. Now X, Y we assumed non-empty, let $x \in X$ $y \in Y$. For any $\lambda \in R_+$ we then have:

$$\phi(x) + \lambda\phi(x_0) \leq \phi(x + \lambda x_0) \leq (x + \lambda x_0)Ay \leq$$

$$\leq (x + \lambda x_0)A(y + \lambda y_0) \leq xA(y + \lambda y_0)$$

$$\leq \psi(y + \lambda y_0) \leq \psi(y) + \lambda\psi(y_0) .$$

Thus, $\psi(y) - \phi(x) \geq \lambda[\phi(x_0) - \psi(y_0)] \geq \lambda \alpha$ for all $\lambda \in R_+$ which contradicts $\alpha > 0$. Thus $0 \in K_1$ and (2) holds. q. e. d.

Proof of Theorems 4 and 5: We prove Theorem 4, the proof of Theorem 5 is analogous. By hypothesis X is non-empty and ϕ is bounded above on X, let:

$$M = \sup_{x \in X} \phi(x)$$

If $(x, \lambda) = (0, M)$ is in K_3 then, trivially, (2) holds. We show that the contrary assumption leads to a contradiction. If $(0, M) \notin K_3$ then, as in the proof of Theorem 4, it follows from the various properties of K_3 that there exist $x_0 \in R^m_+$, $\lambda_0 \in R_+$ and $\alpha \in R$ such that:

$$\lambda_0 M < \alpha \leq x_0(t - Ay) + \lambda_0\psi(y), \quad \text{all } (t, y) \ T \times R^n_+$$

Hence, as before,

$$x_0 Ay \leq \lambda_0\psi(y), \quad \text{all } y \in R^n_+$$

and $\phi(x_0) > \lambda_0 M$.

If λ_0 is positive then $x = \lambda^{-1}x_0 \epsilon X$ and $\phi(x) > M$, contradicting the definition of M. Thus $\lambda_0 = 0$, and $x_0A \leq 0$, $\phi(x_0) > 0$; the last contradicts the fact that X is non-empty and that ϕ is bounded above on X. q.e.d.

As a final result we demonstrate that if ϕ and ψ are both linear (homogeneous) then K_1 is closed. That K_2 and K_3 are closed, under the same linearity hypothesis, follows in a similar manner.

Suppose $\phi(x) = bx$, $\psi(y) = ay$ $(b \epsilon R^m, a \epsilon R^n)$, first note that in this case:

$$S = \left\{ s \mid s \epsilon R^n \text{ and } s \leq a \right\}$$

$$T = \left\{ t \mid t \epsilon R^m \text{ and } t \geq b \right\} .$$

Next, suppose we have a sequence $(\bar{x}_k, \bar{y}_k, \lambda_k) \epsilon K_1$ $(k = 1, 2, \ldots)$ which converges to $(\bar{x}, \bar{y}, \lambda) \epsilon R^{m+n+1}$. Thus there exist $(s_k, t_k, x_k, y_k) \epsilon S \times T \times R_+^m \times R_+^n$ such that:

$$\begin{aligned} &\bar{x}_k \geq t_k - Ay_k \geq b - Ay_k \\ (9) \qquad &\bar{y}_k \leq s_k - x_kA \leq a - x_kA \qquad\qquad k = 1, 2, \ldots \\ &\lambda_k \leq bx_k - ay_k \end{aligned}$$

and

$$(10) \qquad \bar{x}_k \to \bar{x}, \quad \bar{y}_k \to \bar{y}, \quad \lambda_k \to \lambda .$$

Now, suppose $x \epsilon R_+^m$, $y \epsilon R_+^n$, $\alpha \epsilon R_+$ are such that $Ay - \alpha b \geq 0$, $xA - \alpha a \leq 0$. From (9) it then follows that for each k we have:

$$x\bar{x}_k \geq bx - xAy_k \geq bx - \alpha a y_k \geq bx - \alpha b x_k + \alpha\lambda_k \geq$$
$$\geq bx + \alpha\lambda_k - x_kAy \geq bx + \alpha\lambda_k + \bar{y}_k y - ay .$$

i.e., $x(\bar{x}_k - b) + y(a - \bar{y}_k) - \alpha\lambda_k \geq 0, \quad k = 1, 2, \ldots$ and, by (10), $x(\bar{x} - b) + y(a - \bar{y}) - \alpha\lambda \geq 0$.

In summary, then, the system:

$$x \in R_+^m, \quad y \in R_+^n, \quad \alpha \in R_+$$

$$Ay - \alpha b \geq 0, \quad xA - \alpha a \leq 0$$

$$x(\bar{x} - b) + y(a - \bar{y}) - \alpha\lambda < 0$$

has no solution. It follcws then from the ordinary feasibility theorm for linear inequalities (see e.q.[5]) that there is an $x \in R_+^m$ and $y \in R_+^n$ such that:

$$(x, y) \begin{bmatrix} A & 0 & -b \\ 0 & -A^T & a \end{bmatrix} \leq (a - \bar{y}, \; \bar{x} - b, \; -\lambda),$$

i.e., $xA \leq a - \bar{y}$, $-Ay \leq \bar{x} - b$ and $ay - bx \leq -\lambda$. But, as noted before, $a \in S$ and $b \in T$, thus $(\bar{x}, \bar{y}, \lambda) \in K$, and K_1 is closed.

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